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Majorana representation of A_6 involving $3C$-algebras

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Abstract We study a possible Majorana representation $\mathcal{R}$ of the alternating group A_6 of degree 6 such that for some involutions s and t in A_6, generating a D_6-subgroup, the corresponding Majorana axes a_s and a_t generate a subalgebra of type $3C$. We show that there exists at most one such representation $\mathcal{R}$ and that its dimension is at most 70. The representation $\mathcal{R}$ does not correspond to a subalgebra in the Monster algebra generated by a subset of the Majorana axes canonically indexed by the involutions of an A_6-subgroup in the Monster.

1 Majorana representations

A tuple

$$\mathcal{R} = (G, T, V, (\ ,\ ),\ \cdot, \varphi, \psi)$$

is said to be a *Majorana representation* (of G) if the following conditions hold: G is a finite group; T is a set of involutions (elements of order 2) in G which generates G and is stable under conjugation by elements of G (this means that T is a generating union of conjugacy classes of involutions in G); V is a real vector space equipped with an inner product $(\ ,\ )$ and with a commutative non-associative algebra product $\cdot$, which associate with each other and satisfy the *Norton inequality*: $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$ for all $u, v \in V$;

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$$\varphi : G \to GL(V)$$

is a homomorphism, whose image preserves $(\ ,\ )$ and $\cdot$; and ψ is an rule which injectively assigns to every $t \in T$ a vector $a_t = \psi(t)$ which is a *Majorana axis* in $(V, (\ ,\ ), \cdot)$ (cf. the definition in the next paragraph) and such that the Majorana involution associated with a_t coincides with $\varphi(t)$. It is further assumed that V is generated by the elements a_t taken for all $t \in T$, and $\dim(V)$ is said to be the *dimension* of $\mathcal{R}$. It is assumed that φ and ψ commute in the sense that

$$a_{g^{-1}tg} = (a_t)^{\varphi(g)}$$

for every $g \in G$. For a subset X of vectors in V we denote the linear span of X in V and the algebra closure of X in $(V, \cdot)$ by

$$\langle X \rangle \text{ and } \langle\langle X \rangle\rangle,$$

respectively. Thus $\langle\langle X \rangle\rangle$ is the smallest subspace Y in V containing X such that $y_1 \cdot y_2 \in Y$ whenever $y_1, y_2 \in Y$.

By the definition in [1] a Majorana axis a in $(V, (\ ,\ ), \cdot)$ is an idempotent of length 1, whose adjoint operator

$$\mathrm{ad}_a : v \mapsto a \cdot v$$

is semi-simple with spectrum $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ (when talking about eigenvectors of a one really means eigenvectors of ad_a). The following conditions concerning the eigenspaces are imposed. The 1-eigenvectors of a are precisely the scalar multiples of a. The Majorana involution $\tau(a)$ associated with a is the linear transformation of V which negates every $\frac{1}{32}$-eigenvector and centralizes the other eigenvectors. By the above definition the Majorana involution is an automorphism of $(V, (\ ,\ ), \cdot)$. This condition is equivalent to the *fusion rules* involving $\frac{1}{32}$-eigenvectors: if v and u are $\frac{1}{32}$-eigenvectors of a, and if x and y are λ- and μ-eigenvectors for $\lambda, \mu \in \{1, 0, \frac{1}{4}\}$, then $v \cdot x$ is a $\frac{1}{32}$-eigenvector, and both $v \cdot u$ and $x \cdot y$ project to zero in the $\frac{1}{32}$-eigenspace. The remaining fusion rules are as follows: if α_1 and α_2 are 0-eigenvectors of a, and β_1 and β_2 are $\frac{1}{4}$-eigenvectors of a, then

$$\alpha_1 \cdot \alpha_2, \quad \beta_1 \cdot \beta_2 - (\beta_1 \cdot \beta_2, a)\, a \quad \text{and} \quad \alpha_1 \cdot \beta_1$$

are λ-eigenvectors of a, where $\lambda = 0,\ 0$ and $\frac{1}{4}$, respectively. The complete set of fusion rules can be read from Table 1 (where $Sp = \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$ is the spectrum of a). The meaning of the fusion rules is the inclusion

$$V_\lambda^{(a)} \cdot V_\mu^{(a)} \subseteq \bigoplus_{\nu \in Sp(\lambda,\mu)} V_\nu^{(a)}$$

where $\lambda, \mu \in Sp$ and $Sp(\mu, \lambda)$ is the (λ, μ)-entry in Table 1.

Table 1 Fusion rules

Sp	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	0	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

Sakuma's theorem [9], together with Norton's explicit description of the subalgebras in the Monster algebra generated by pairs of transposition axes [7], implies the following proposition, which constitutes the foundation of the Majorana theory.

Proposition 1.1 [7,9] *Let $\mathcal{R} = (G, T, V, (\ ,\ ), \cdot, \varphi, \psi)$ be a Majorana representation. For two distinct involutions t_0 and t_1 in T put $a_0 = \psi(t_0), a_1 = \psi(t_1), \tau_0 = \varphi(t_0), \tau_1 = \varphi(t_1), \rho = t_0t_1$. Let D be the dihedral subgroup in $GL(V)$ generated by τ_0 and τ_1, and let $|D| = 2N$. Then the subalgebra $Y = \langle\langle a_0, a_1\rangle\rangle$ is isomorphic to one of the eight Norton–Sakuma algebras in Table 2 (more specifically to an algebra of type NX, where N is as above and $X \in \{A, B, C\}$). For an integer i and $\varepsilon \in \{0, 1\}$ the vector $a_{2i+\varepsilon}$ is the image of a_ε under the i-th power of ρ (so that $\tau(a_{2i+\varepsilon}) = \rho^{-i}\tau_\varepsilon\rho^i$), and the remaining vectors in the basis of Y, given in the second column in Table 2, are centralized by D. The kernel of the action of D on Y coincides with the centre of D.*

The *shape* of a Majorana representation $\mathcal{R}$ is a rule which specifies the type of the Norton–Sakuma subalgebra $\langle\langle\psi(t_0), \psi(t_1)\rangle\rangle$ for every pair t_0, t_1 of involutions in T. The rule must be stable under conjugation by the elements of G and must respect the embeddings of the algebras:

$$2A \hookrightarrow 4B,\ \ 2A \hookrightarrow 6A,\ \ 2B \hookrightarrow 4A,\ \ 3A \hookrightarrow 6A.$$

The Monster algebra possesses important properties which we are included sometimes into the hypothesis:

(2A) the following conditions hold (where $t_0, t_1, t_2 \in T$ and $a_i = \psi(t_i)$ for $0 \le i \le 2$):
- (a) if $t_0t_1t_2 = 1$, then $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$ and $a_2 = a_\rho := a_0 + a_1 - 8a_0 \cdot a_1$;
- (b) if $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$, $4B$ or $6A$, then t_0t_1, $(t_0t_1)^2$ or $(t_0t_1)^3$ belongs to T, and $\psi(t_0t_1)$, $\psi((t_0t_1)^2)$ or $\psi((t_0t_1)^3)$ coincides with a_ρ, a_{ρ^2} or a_{ρ^3}.

(3A) the following condition holds (where $t_0, t_1, t_2, t_3 \in T$, $a_i = \psi(t_i)$ for $0 \le i \le 3$): if
- (a) $\langle t_0, t_1\rangle \cong \langle t_2, t_3\rangle \cong D_6$;
- (b) $t_0t_1 = t_2t_3$;
- (c) both $\langle\langle a_0, a_1\rangle\rangle$ and $\langle\langle a_2, a_3\rangle\rangle$ have type $3A$;

then the $3A$-axial vectors $u_{t_0t_0}$ and $u_{t_2t_3}$ in the subalgebras in (c) are equal.

Table 2 Norton–Sakuma algebras

Type	Basis	Products and angles
2A	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho),\ a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$
		$(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{2^3}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0, (a_0, a_1) = 0$
3A	$a_{-1}, a_0, a_1,$	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3.5}{2^{11}} u_\rho$
	u_ρ	$a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$
		$u_\rho \cdot u_\rho = u_\rho$
		$(a_0, a_1) = \frac{13}{2^8}, (a_0, u_\rho) = \frac{1}{2^2}, (u_\rho, u_\rho) = \frac{2^3}{5}$
3C	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1}),\ (a_0, a_1) = \frac{1}{2^6}$
4A	$a_{-1}, a_0, a_1,$	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v_\rho)$
	a_2, v_ρ	$a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$
		$v_\rho \cdot v_\rho = v_\rho,\ a_0 \cdot a_2 = 0$
		$(a_0, a_1) = \frac{1}{2^5},\ (a_0, a_2) = 0,\ (a_0, v_\rho) = \frac{3}{2^3},\ (v_\rho, v_\rho) = 2$
4B	$a_{-1}, a_0, a_1,$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$
	a_2, a_{ρ^2}	$a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$
		$(a_0, a_1) = \frac{1}{2^6},\ (a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{2^3}$
5A	$a_{-2}, a_{-1}, a_0,$	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$
	a_1, a_2, w_ρ	$a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$
		$a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$
		$w_\rho \cdot w_\rho = \frac{5^2.7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$
		$(a_0, a_1) = \frac{3}{2^7},\ (a_0, w_\rho) = 0, (w_\rho, w_\rho) = \frac{5^3.7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0,$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2.5}{2^{11}} u_{\rho^2}$
	a_1, a_2, a_3	$a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3.5}{2^{11}} u_{\rho^2}$
	a_{ρ^3}, u_{ρ^2}	$a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$
		$a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0, (a_{\rho^3}, u_{\rho^2}) = 0$
		$(a_0, a_1) = \frac{5}{2^8}, (a_0, a_2) = \frac{13}{2^8}, (a_0, a_3) = \frac{1}{2^3}$

2 The alternating group A_6

Let $G \cong A_6$. Then G contains a single class of involutions, two classes of subgroups isomorphic to the alternating group A_5 of degree 5 (say $\mathcal{K}_1$ and $\mathcal{K}_2$), two classes of subgroups of order 3 (say $H^{(1)}$ and $H^{(2)}$), one class of subgroups of order 5, and two classes of elements of order 5. Let $K \in \mathcal{K}_1$ be an A_5-subgroup in G and let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the set of cosets of K in G. We assume that the subgroups in $H^{(1)}$ are generated by 3-cycles on Ω and those in $H^{(2)}$ by products of two commuting 3-cycles. Thus a Majorana representation of A_6 satisfying conditions (2A) and (3A) has shape $(2A, 3X, 3Y, 4B, 5A)$, where

$$\langle\langle a_s, a_t\rangle\rangle \cong 3X \quad \text{or } 3Y \quad \text{if } \langle s, t\rangle \in H^{(1)} \quad \text{or } H^{(2)},$$

respectively. Since the classes $H^{(1)}$ and $H^{(2)}$ are fused in the automorphism group of A_6, a representation of shape $(2A, 3X, 3Y, 4B, 5A)$ twisted by a suitable automorphism of A_6 has shape $(2A, 3Y, 3X, 4B, 5A)$.

It was proved in [2] that A_6 possesses a unique representation of shape $(2A, 3A, 3A, 4B, 5A)$, which satisfies conditions (2A) and (3A). Here we prove the following.

Theorem 1 *The following assertions hold:*

(i) *there are no Majorana representations of* A_6 *of shape* $(2A, 3A, 3C, 4B, 5A)$ *satisfying condition* (2A)*;*

(ii) *there exists at most one Majorana representation* $\mathcal{R}^{CC}$ *of* A_6*, satisfying condition* (2A)*, whose shape is* $(2A, 3C, 3C, 4B, 5A)$*; the dimension of* $\mathcal{R}^{CC}$ *is at most* 70.[1]

3 Inner products

In this section by $\mathcal{R}^{AC}$ or $\mathcal{R}^{CC}$ we denote a representation satisfying the hypothesis in Theorem 1 (i) or (ii), so that

$$\langle A^2\rangle = \langle A \cup U^{(1)} \cup W\rangle \ \text{ or } \langle A^2\rangle = \langle A \cup W\rangle,$$

respectively, where $A = \{a_t \mid t^2 = 1\}$, $U^{(1)} = \{u_h \mid \langle h\rangle \in H^{(1)}\}$, $W = \{w_f \mid f^5 = 1\}$ with $u_{h^{-1}} = u_h$, $w_f = -w_{f^2} = -w_{f^3} = w_{f^4}$.

Lemma 3.1 *Let* $\langle h\rangle, \langle k\rangle \in H^{(1)}$*. If* $[h, k] = 1$ *then in the representation* $\mathcal{R}^{AC}$ *the inner product* (u_h, u_k) *is negative.*

Proof For $h = (1, 2, 3)$, $k = (4, 5, 6)$ and $t = (2, 3)(4, 5)$ the vector

$$\alpha_h^{(t)} = u_h - \frac{2\cdot 5}{3^3}a_t + \frac{2^5}{3^3}\left(a_{hth^{-1}} + a_{h^{-1}th}\right)$$

is a 0-eigenvector of a_t in $\langle\langle a_t, u_h\rangle\rangle \cong 3A$, and

$$\beta_k^{(t)} = u_k - \frac{2^3}{3^2\cdot 5}a_t - \frac{2^5}{3^2\cdot 5}\left(a_{ktk^{-1}} + a_{k^{-1}tk}\right)$$

is a $\frac{1}{4}$-eigenvector of a_t in $\langle\langle a_t, u_k\rangle\rangle \cong 3A$. Expanding the orthogonality relation $(\alpha_h^{(t)}, \beta_k^{(t)}) = 0$ one gets

[1] 70-dimensional representation $\mathcal{R}^{CC}$ was constructed by Ákos Seress (private communication of February 9, 2011).

Table 3 Inner products

	$o(rf)$	r	$\langle r, f\rangle$	(a_r, w_f)
1	2	$(2,5)(3,4)$	D_{10}	0
2	3	$(2,3)(4,5)$	$A_5 \in \mathcal{K}_1$	$\frac{7}{2^{13}}$
3	5	$(2,4)(3,5)$	$A_5 \in \mathcal{K}_1$	$-\frac{7}{2^{13}}$
4	5	$(1,6)(3,4)$	$A_5 \in \mathcal{K}_2$	$-\frac{7}{2^{13}}$
5	3	$(1,6)(2,5)$	$A_5 \in \mathcal{K}_2$	$\frac{7}{2^{13}}$
6	4	$(1,6)(3,5)$	A_6	0
7	5	$(1,6)(2,3)$	A_6	0

$$(u_h, u_k) = \frac{2^4}{3^5 \cdot 5}(5 - 18 + 4),$$

and the result follows. □

Thus the Norton inequality condition fails under the hypothesis of Theorem 1 (i) and from now on we only deal with the representation $\mathcal{R}^{CC}$ (where all 3-elements are of type $3C$). The Majorana representations of A_5 were classified in [5]. There only one such representation which satisfying the $(2A)$ condition has shape $(2A, 3C, 5A)$. Thus the restriction of $\mathcal{R}^{CC}$ to an A_5-subgroup is known.

Recall that $\operatorname{Out}(A_6)$ is elementary abelian of order 4, and if $A_6 < X < \operatorname{Aut}(A_6)$, then $X \cong S_6$, $PGL_2(9)$ or M_{10}. In S_6 all the elements of order 5 are conjugate, but there are still two classes of elements/subgroups of order 3; in $PGL_2(9)$ there are two classes of elements of order 5, but all elements of order 3 are conjugate, while M_{10} enjoys both the conjugation properties: a single class of 5-elements and a single class of 3-elements.

The following lemma is a direct consequence of the Majorana axioms and the shape of $\mathcal{R}^{CC}$.

Lemma 3.2 *If s and t are involutions in A_6, then $(a_s, a_t) = 1,\ \frac{1}{8},\ \frac{1}{64},\ \frac{1}{64}$ or $\frac{3}{128}$ if $o(st) = 1,\ 2,\ 3,\ 4$ or 5, respectively.* □

Lemma 3.3 *Let r be an involution in A_6 and let f be an element of order 5 in A_6. Then the possible values of (a_r, w_f) are described by Table 3, where $f = (1, 2, 3, 4, 5)$.*

Proof Since the restriction of $\mathcal{R}^{CC}$ to an A_5-subgroup is known, it is sufficient to justify the zero entries in the last column of the last two rows. For the sixth row consider $t = (2, 5)(3, 4)$. Then t inverts f and generates with r a D_8-subgroup. Therefore

$$\beta_f^{(t)} = w_f + \frac{1}{2^7}(-a_{ftf^4} + a_{f^2tf^3} + a_{f^3tf^2} - a_{f^4tf})$$

(which is a $\frac{1}{4}$-eigenvector of a_t in $\langle\langle a_t, w_f\rangle\rangle \cong 5A$) and

$$\alpha_r^{(t)} = a_r + a_{trt} - \frac{1}{2^5}a_t - \frac{1}{2^3}(a_{(rt)^2} - a_{rtr})$$

Table 4 Order 5 subgroups in A_6

(1, 2, 3, 4, 5)	(1, 4, 5, 2, 3)	(1, 2, 4, 5, 3)	(1, 2, 5, 3, 4)	(1, 2, 5, 3, 4)	(1, 5, 2, 4, 3)
(6, 5, 3, 4, 2)	(6, 3, 5, 2, 4)	(6, 5, 4, 2, 3)	(6, 4, 5, 3, 2)	(6, 5, 2, 3, 4)	(6, 2, 5, 4, 3)
(6, 1, 4, 5, 3)					
(6, 2, 5, 1, 4)					
(6, 3, 1, 2, 5)					
(6, 4, 2, 3, 1)					

(which is a 0-eigenvector of a_t in $\langle\langle a_t, a_r\rangle\rangle \cong 4B$) are perpendicular. Since t stabilizes w_f and swaps a_r with a_{trt}, we have

$$(w_f, a_r) = (w_f, a_{trt}).$$

Now, making use of Lemma 3.1 and substituting the known entries from the last columns of the second and third rows in Table 3, we deduce the equality $(w_f, a_r) = 0$. For the seventh row the calculations are similar. □

Next we determine the inner products between w_f's. Before doing that we describe the A_6-orbits on the pairs of its subgroups of order 5. The set of these subgroups forms a 6×6 grid with rows corresponding to the A_5-subgroups in $\mathcal{K}_1$ and columns corresponding to the A_5-subgroups in $\mathcal{K}_2$. Every subgroup of $F = \langle f\rangle$ of order 5 in A_6 is contained in a unique subgroup $K_1(f) \in \mathcal{K}_1$ and in a unique subgroup $K_2(f) \in \mathcal{K}_2$, so that

$$K_1(f) \cap K_2(f) = N_{A_6}(F) \cong D_{10}.$$

The following (partially filled up) Table 4 contains generators of the subgroups of order 5 in two members of $\mathcal{K}_1$ (the first and second rows) and in two members of $\mathcal{K}_2$ (the first and the second columns). The generators are chosen to be A_6-conjugate, and the subgroup generated by the element in the i-th row and j-th column will be denoted by F_{ij}, so that $F_{11} = \langle(1, 2, 3, 4, 5)\rangle$.

Lemma 3.4 *Let t be an involution in A_6. Then the following assertions hold:*

(i) *t is contained in exactly two members of $\mathcal{K}_1$ and in two members of $\mathcal{K}_2$;*
(ii) *t normalizes exactly four subgroups of order 5 in A_6;*
(iii) *if F and E are two distinct subgroups of order 5 normalized by t then*
 (a) *there is an element $s \in A_6$ such that $E = s^{-1}Fs$ and $[s, t] = 1$;*
 (b) *$s^2 = 1$ if $\langle E, F\rangle \cong A_5$ and $s^4 = 1 \neq s^2$ otherwise;*
(iv) *if F is a subgroup of order 5 and K is an A_5-subgroup not containing F, then K contains a unique involution which normalizes F.*

Proof These are all very elementary and easy to check, especially making use of the outer automorphism of A_6 which permutes $\mathcal{K}_1$ and $\mathcal{K}_2$. For instance, if $t = (2, 5)(3, 4)$, then the A_5-subgroups in (i) containing t correspond to the first and the second row/column Table 4; the subgroups in (ii) normalized by t are F_{11}, F_{12}, F_{21} and F_{22}. Since

$N_{A_6}(t) \cong D_8$ and for $t \in A_5$ we have $N_{A_5}(t) \cong 2^2$, (iii) follows. The pair $\{F_{11}, F_{22}\}$ provides an example of two subgroups of order 5 normalized by a common involution, but not contained in a common A_5-subgroup. In (iv) if $F = F_{11}$ and K corresponds to the second row (so that K is the stabilizer in A_6 of $1 \in \Omega$) then $t = (2,5)(3,4)$ is the unique involution with the required properties. $\square$

Lemma 3.5 *Let f and e be elements of order 5 in A_6 contained in the same conjugacy class, such that $F = \langle f \rangle$ and $E = \langle e \rangle$ are distinct. Then the following assertions hold:*

(i) $(w_f, w_f) = \frac{5^3 \cdot 7}{2^{19}}$;
(ii) *if* $\langle F, E \rangle \cong A_5$, *then* $(w_f, w_e) = -\frac{5^2 \cdot 7}{2^{19}}$;
(iii) *if* $\langle F, E \rangle = A_6$, *then* $(w_f, w_e) = 0$ *whenever f and e are inverted by a common involution, and* $(w_f, w_e) = \frac{5^2 \cdot 7}{2^{21}}$, *otherwise.*

In order to prove the above lemma we will make use of the following restatement of Lemma 5.4 from [5].

Lemma 3.6 *Let e be an element of order 5 in $K \cong A_5$ and t be an involution in K which does not normalize $\langle e \rangle$. Then*

$$a_t \cdot w_e = (a_t, w_e)\, a_t - \frac{7}{2^7}(\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}) + \frac{1}{2^6}(w_e - w_{tet}),$$

where f_1 and f_2 are K-conjugates of e inverted by t and generating distinct subgroups, and for $d = f_i$ for $i = 1$ or 2

$$\beta^{(t)}_d = w_d + \frac{1}{2^7}(-a_{dtd^4} + a_{d^2td^3} + a_{d^3td^2} - a_{d^4td})$$

is a $\frac{1}{4}$-eigenvector of a_t in $\langle\langle a_t, w_f \rangle\rangle \cong 5A$. $\square$

An immediate consequence of Lemma 3.6 and the Majorana axioms is the following.

Lemma 3.7 *With e and t being as in Lemma 3.6 the projections of w_e to 1-, $\frac{1}{32}$-, $\frac{1}{4}$- and 0-eigenspaces of a_t are equal to*

$$(a_t, w_e)\, a_t, \quad \frac{1}{2}(w_e - w_{tet}), \quad -\frac{7}{2^5}(\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}), \quad \text{and}$$

$$\varepsilon^{(t)}_e := \frac{1}{2}(w_e + w_{tet}) - (a_t, w_e)\, a_t + \frac{7}{2^5}(\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}),$$

respectively. $\square$

Now suppose that $f \notin K$. Then by Lemma 3.4 (iv) we can assume without loss of generality that f is inverted by t, which provides us with an algorithm of calculating (w_f, w_e) from the orthogonality relation

$$(\varepsilon_e^{(t)}, \beta_f^{(t)}) = 0.$$

In what follows we will demonstrate an easier method how to get (w_f, w_e). At this stage we just summarize an important consequence of the orthogonality relation.

Lemma 3.8 *The inner product (w_e, w_f) is the same whenever e and f are conjugate and not inverted by a common involution.* □

An important property of the Majorana representation of A_5 of shape $(2A, 3C, 5A)$ is the following (cf. Lemma 5.7 (i) in [5]).

Lemma 3.9 *Let $K^{(5)}$ be a set of conjugate representatives of the subgroups of order 5 in $K \cong A_5$. Then*

$$\sum_{f \in K^{(5)}} w_f = 0.$$

□

Proof of Lemma 3.7. The assertions (i) and (ii) follow from Lemmas 4.1 (vii) and 4.2 (ii) in [5], respectively. Let e and f be as in (iii). If t is an involution which simultaneously inverts e and f, then the value of (w_f, w_e) can be deduced from the orthogonality relation

$$(\alpha_e^{(t)}, \beta_f^{(t)}) = 0,$$

where $\beta_f^{(t)}$ is the a $\frac{1}{4}$-eigenvector of a_t in $\langle\langle a_t, w_f\rangle\rangle \cong 5A$ defined in Lemma 3.6, while

$$\alpha_e^{(t)} = w_e - \frac{3 \cdot 7}{2^{12}} a_t + \frac{7}{2^6}(a_{ete^4} + a_{e^4te})$$

is a 0-eigenvector of a_t in $\langle\langle a_t, w_e\rangle\rangle \cong 5A$ (cf. Table 3 in [5]). Thus it only remains to handle the case when f and e are not inverted by a common involution. Let K be an A_5-subgroup in A_6, let $K^{(5)}$ be as in Lemma 3.9, and let e an element of order 5, which is not in K, but conjugate in A_6 to members of $K^{(5)}$. Then by Lemma 3.4 $K^{(5)}$ contains a unique element contained in a common A_5-subgroup with e and a unique element not in a common A_5-subgroup with e, but simultaneously with e inverted by an involution and four elements in the 'general position' with respect to e. Thus if $\lambda = (w_a, w_b)$ where a and b are in the general position, then by Lemmas 3.8 and 3.9 we obtain

$$0 = \left(w_e, \sum_{f \in K^{(5)}} w_f \right) = -\frac{5^2 \cdot 7}{2^{19}} + 0 + 4\lambda,$$

which gives $\lambda = \frac{5^2 \cdot 7}{2^{21}}$ as claimed. □

4 Product closure

For a subgroup H in A_6 put $A(H) = \{a_t \mid t \in H, t^2 = 1\}$, $W(H) = \{w_f \mid f \in H, f^5 = 1\}$, and let $X(H)$ denote the linear span of $A(H) \cup W(H)$. Our goal is to prove that the algebra product $\cdot$ is closed on $X(A_6)$. Since $\cdot$ is closed on $X(K)$ for every A_5-subgroup K in A_6, in order to achieve the goal it is sufficient to prove the following two assertions.

Proposition 4.1 *$X(A_6)$ contains $a_r \cdot w_f$ whenever $\langle r, f\rangle = A_6$.*

Proposition 4.2 *$X(A_6)$ contains $w_f \cdot w_e$ whenever $\langle f, e\rangle = A_6$.*

4.1 Proof of Proposition 4.1

Throughout the subsection we assume that $f = (1, 2, 3, 4, 5)$ and $r = (1, 6)(3, 5)$, so that the pair (f, r) corresponds to the sixth row of Table 3. The situation when the pair corresponds to the seventh row can be handled in a similar way.

There are five involutions in A_6 inverting f. If s_j denotes the involution which inverts f and stabilizes the element $j \in \Omega$, then the information on the relationship between r and s_j can be read from the following table.

For every $1 \le j \le 5$ the subalgebra $Z_j = \langle\langle w_f, a_{s_j}\rangle\rangle$ is of type $5A$. Now for j being 4 or 5 we apply a variation of the resurrection principle (cf. Lemma 1.8 in [3]). The vectors

$$\alpha_f^{(s_j)} = w_f - \frac{3 \cdot 7}{2^{12}} a_{s_j} + \frac{7}{2^6}(a_{f s_j f^4} + a_{f^4 s_j f})$$

and

$$\beta_f^{(s_j)} = w_f + \frac{1}{2^7}(-a_{f s_j f^4} + a_{f^2 s_j f^3} + a_{f^3 s_j f^2} - a_{f^4 s_j f})$$

are 0- and $\frac{1}{4}$-eigenvectors of a_{s_j} in Z_j, while

$$\alpha_r^{(s_j)} = (a_r + a_{s_j r s_j}) - \frac{1}{32} a_{s_j}$$

is a 0-eigenvector of a_{s_j} in $Y_j \cong 3C$.

Consider

$$\alpha^{(s_j)} := \alpha_f^{(s_j)} \cdot \alpha_r^{(s_j)} = (a_r + a_{s_jrs_j}) \cdot w_f + x,$$
$$\beta^{(s_j)} := \beta_f^{(s_j)} \cdot \alpha_r^{(s_j)} = (a_r + a_{s_jrs_j}) \cdot w_f + y,$$

which are 0- and $\frac{1}{4}$-eigenvectors of a_{s_j} and where the vectors x and y are contained in $X(A_6)$, since $X(A_6)$ contains the product $a_t \cdot a_s$ for all $t, s \in T$. By the resurrection principle Lemma 1.8 in [3] we have

$$(a_r + a_{s_jrs_j}) \cdot w_f = -[4a_{s_j} \cdot (x - y) + y].$$

This gives the following (notice that $w_f^{\varphi(s_j)} = w_{f^{-1}} = w_f$).

Lemma 4.3 *For $j = 4$ and 5 the vector $a_r \cdot w_f$ and the negative of $(a_r \cdot w_f)^{\varphi(s_j)} = a_{s_jrs_j} \cdot w_f$ are equal modulo $X(A_6)$.* □

It is easy to prove the above lemma also for $j = 1$ and 3, but we don't need it.

Lemma 4.4 *Modulo $X(A_6)$ the vectors $a_r \cdot w_f$ and $(a_r \cdot w_f)^{\varphi(f^2)} = a_{f^3rf^2} \cdot w_f$ are equal.* □

Proof Multiplying s_4 by s_5 in A_6 we obtain f^2, and since $(-1)(-1) = 1$, the result follows. □

The vector $w_f - w_{rfr}$ is a $\frac{1}{32}$-eigenvector of a_r:

$$a_r \cdot (w_f - w_{rfr}) = \frac{1}{32}(w_f - w_{rfr}).$$

Since the right hand side of the above equality in contained in $X(A_6)$, this gives

Lemma 4.5 *The vectors $a_r \cdot w_f$ and $(a_r \cdot w_f)^{\varphi(r)} = (a_r \cdot w_{rfr})$ are equal modulo $X(A_6)$.* □

Since $\langle r, f \rangle = A_6$, Lemmas 4.4 and 4.5 imply that

$$(a_r \cdot w_f)^{\varphi(g)} \equiv a_r \cdot w_f \mod X(A_6)$$

for every $g \in A_6$. Combining this equality taken for $g = s_j$ with Lemma 4.3 we observer that $a_r \cdot w_f$ is equal to its negative modulo $X(A_6)$ which completes the proof of Proposition 4.1.

4.2 Proof of Proposition 4.2

The core of the argument here is similar to that in the previous subsection. Notice that $X(A_6)$ contains all the products $a_t \cdot a_s$ for $t, s \in T$ by definition and all the products $a_t \cdot w_f$ by Proposition 4.1 and since the product $\cdot$ is closed on $X(K)$ for every A_5-subgroup K in A_6.

Let f and e be two conjugate elements of order 5 in A_6, generating the whole of A_6. Suppose first both f and e are inverted by an involution t. In this case the product $w_f \cdot w_e$ can calculated by applying the resurrection principle to the 0-eigenvector

$$\alpha_f^{(t)} = w_f - \frac{3 \cdot 7}{2^{12}} a_t + \frac{7}{2^6}(a_{ftf^4} + a_{f^4tf})$$

of a_t in $\langle\langle a_t, w_f \rangle\rangle \cong 5A$ and the $\frac{1}{4}$-eigenvector

$$\beta_e^{(t)} = w_f + \frac{1}{2^7}(-a_{ftf^4} + a_{f^2tf^3} + a_{f^3tf^2} - a_{f^4tf})$$

of a_t in $\langle\langle a_t, w_e \rangle\rangle \cong 5A$. Therefore from now on we assume that f and e are not inverted by a common involution.

Let $K_i(e)$ be the unique A_5-subgroup in $\mathcal{K}_i$ containing e and let $t_i(f, e)$ be the unique involution in $K_i(e)$ which inverts t (cf. Lemma 3.4 (iv)). Put

$$t_1 = t_1(f, e), \quad t_2 = t_2(f, e), \quad t_3 = t_1(e, f), \quad t_4 = t_2(e, f).$$

Lemma 4.6 *The involutions t_1, t_2, t_3 and t_4 are pairwise distinct, and*

$$\langle t_1t_2 \rangle = \langle f \rangle, \quad \langle t_3t_4 \rangle = \langle e \rangle.$$

Proof Since $K_1(e) = K_1(e^{t_1})$ and $K_2(e) = K_2(e^{t_2})$, the equality $t_1 = t_2$ would immediately imply that t_1 inverts e. In a similar way one shows that $t_3 \neq t_4$. It is also clear that $t_j \neq t_k$ for $1 \leq j \leq 2$ and $3 \leq k \leq 4$ since t_j inverts f, while t_k inverts e, while f and e are chosen not to be inverted by a common involution. Since $t_1, t_2 \in N_{A_6}(\langle f \rangle) \cong D_{10}$ and $t_3, t_4 \in N_{A_6}(\langle e \rangle) \cong D_{10}$ the equalities in the assertion follow. □

For $i = 1$ or 2 let $\varepsilon_e^{(t_i)}$ be the 0-eigenvector of a_{t_i} defined in Lemma 3.7 and contained in the image of the restriction of $\mathcal{R}^{CC}$ to $K_i(e) \cong A_5$. Let $\alpha_f^{(t_i)}$ and $\beta_f^{(t_i)}$ be the 0- and $\frac{1}{4}$-eigenvectors of a_{t_i} as in the paragraph after Table 5 (with s_j substituted by t_i) which are contained in $\langle\langle a_{t_i}, w_f \rangle\rangle \cong 5A$. By the fusion rules we have

$$\alpha := \alpha_f^{(t_i)} \cdot \varepsilon_e^{(t_i)} = w_f \cdot (w_e + w_{t_iet_i}) + u,$$
$$\beta := \beta_f^{(t_i)} \cdot \varepsilon_e^{(t_i)} = w_f \cdot (w_e + w_{t_iet_i}) + v$$

are 0-and $\frac{1}{4}$-eigenvectors of t_i, where u and v are explicitly computable vectors from $X(A_6)$.

Table 5 Neighbouring subalgebras

j	4	5	1	3	2
$o(rs_j)$	3	3	4	4	5
$Y_i := \langle\langle a_r, a_{s_j} \rangle\rangle$	$3C$	$3C$	$4B$	$4B$	$5A$

Lemma 4.7 *The following assertions hold:*

(i) *for* $1 \leq i \leq 4$ *the vector* $w_f \cdot w_e$ *is equal to the negative of* $(w_f \cdot w_e)^{\varphi(t_i)} = w_{t_i f t_i} \cdot w_{t_i e t_i}$ *modulo* $X(A_6)$*;*

(ii) *the vectors* $w_f \cdot w_e$, $(w_f \cdot w_e)^{\varphi(f)} = w_f \cdot w_{f^{-1}ef}$ *and* $(w_f \cdot w_e)^{\varphi(e)} = w_{e^{-1}fe} \cdot w_e$ *are equal modulo* $X(A_6)$.

Proof The assertion (i) for $i = 1$ or 2 follows from the resurrection principle applied to the eigenvectors α and β defined just before the lemma. This assertion for $i = 3$ or 4 is obtained by exchanging the roles of f and e. Now (ii) follows from (i) and Lemma 4.6. □

Since $\langle f, e \rangle = X(A_6)$, it follows from Lemma 4.7 (ii) that

$$(w_f \cdot w_e)^{\varphi(g)} \equiv w_f \cdot w_e \mod X(A_6)$$

for every $g \in A_6$. Combining this equality for $g = t_i$ with Lemma 4.7 (i) we conclude that $w_f \cdot w_e$ equals to its negative modulo $X(A_6)$ and Proposition 4.2 follows.

4.3 Dimension

By Propositions 4.1 and 4.2 the dimension of $\mathcal{R}^{CC}$ equals to the rank of the Gram matrix Γ of the set $A(A_6) \cup W(A_6)$ of size $45+36 = 81$. We are certain that this matrix is singular. In fact, every A_5-subgroup K in A_6 leads to a relation as in Lemma 3.9. There are 12 such subgroup, but there is at least one dependence among these twelve relations: the sums of the relations in Lemma 3.9 over $\mathcal{K}_1$ and $\mathcal{K}_2$ lead to the same equality

$$\sum_{f \in X^{(5)}} w_f = 0,$$

where $X^{(5)}$ is a set of conjugate representatives of subgroups of order 5 in A_6. Therefore we have the following upper bound on the dimension of $\mathcal{R}^{CC}$.

Proposition 4.8 *The dimension of* $\mathcal{R}^{CC}$ *is at most* 70.[2] □

[2] Igor Faradjev has computed the eigenvalues of the Gram matrix Γ. The zero eigenvalue appears with multiplicity 11, which means that the upper bound is attained (private communication of January 4, 2011).

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